

OPTIMAL NOVIKOV-TYPE CRITERIA FOR LOCAL MARTINGALES WITH JUMPS

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ABSTRACT. We consider local martingales M with jumps larger than a for some a larger than or equal to -1 , and prove Novikov-type criteria for an exponential local martingale to be a uniformly integrable martingale. We obtain criteria using both the quadratic variation and the predictable quadratic variation. We prove optimality of the coefficients in the criteria. As a corollary, we obtain a verbatim extension of the Novikov criterion to the case of local martingales with nonnegative jumps.

1. Introduction

The motivation of this paper is the question of when an exponential local martingale is a uniformly integrable martingale. Before introducing this problem, we fix our notation and recall some results from stochastic analysis.

Assume given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ satisfying the usual conditions, see [11] for the definition of this and other probabilistic concepts such as being a local martingale, locally integrable, locally square-integrable, and for the quadratic variation and quadratic covariation et cetera. For any local martingale M , we say that M has initial value zero if $M_0 = 0$. For any local martingale M with initial value zero, we denote by $[M]$ the quadratic variation of M , that is, the unique increasing adapted process with initial value zero such that $M^2 - [M]$ is a local martingale. If M furthermore is locally square integrable, we denote by $\langle M \rangle$ the predictable quadratic variation of M , which is the unique increasing predictable process with initial value zero such that $[M] - \langle M \rangle$ is a local martingale.

For any local martingale with initial value zero, there exists by Theorem 7.25 of [2] a unique decomposition $M = M^c + M^d$, where M^c is a continuous local martingale and M^d is a purely discontinuous local martingale, both with initial value zero. Here, we say that a local martingale with initial value zero is purely discontinuous if it has zero quadratic covariation with any continuous local martingale with initial

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value zero. We refer to M^c as the continuous martingale part of M , and refer to M^d as the purely discontinuous martingale part of M .

With M a local martingale with initial value zero and $\Delta M \geq -1$, the exponential martingale of M , also known as the Doléans-Dade exponential of M , is the unique càdlàg solution in Z to the stochastic differential equation $Z_t = 1 + \int_0^t Z_{s-} dM_s$, given explicitly as

$$(1.1) \quad \mathcal{E}(M)_t = \exp\left(M_t - \frac{1}{2}[M^c]_t\right) \prod_{0 < s \leq t} (1 + \Delta M_s) \exp(-\Delta M_s),$$

see Theorem II.37 of [11]. Applying Theorem 9.2 of [2], we find that Z always is a local martingale with initial value one. We wish to understand when $\mathcal{E}(M)$ is a uniformly integrable martingale.

The question of when $\mathcal{E}(M)$ is a uniformly integrable martingale has been considered many times in the literature, and is not only of theoretical interest, but has several applications in connection with other topics. In particular, exponential martingales are of use in mathematical finance, where checking uniform integrability of a particular exponential martingale can be used to prove absence of arbitrage and obtain equivalent martingale measures for option pricing. For more on this, see [10] or chapters 10 and 11 of [1]. Also, exponential martingales arise naturally in connection with maximum likelihood estimation for stochastic processes, where the likelihood viewed as a stochastic process often is an exponential martingale which is a true martingale, see for example the likelihood for parameter estimation for homogeneous Poisson processes given in (3.43) of [6]. Finally, exponential martingales which are true martingales can be used in the explicit construction of various probabilistic objects, for example solutions to stochastic differential equations, as in Section 5.3.B of [5].

Several sufficient criteria for $\mathcal{E}(M)$ to be a uniformly integrable martingale are known. First results in this regard were obtained by [8] for the case of continuous local martingales. Here, we are interested in the case where the local martingale M is not necessarily continuous. Sufficient criteria for $\mathcal{E}(M)$ to be a uniformly integrable martingale in this case have been obtained by [7], [3], [9], [14] and [4]. [7] appears to contain the first criteria of this type, with [14] containing simplified proofs of several of the results in [7]. Here, we will be applying the results of [7].

We now explain the particular result to be obtained in this paper. In [8], the following result was obtained: If M is a continuous local martingale with initial value zero and $\exp(\frac{1}{2}[M]_\infty)$ is integrable, then $\mathcal{E}(M)$ is a uniformly integrable martingale. This criterion is known as Novikov's criterion. We wish to understand whether this result can be extended to local martingales which are not continuous.

In the case with jumps, another process in addition to the quadratic variation process is relevant: the predictable quadratic variation. As noted earlier, the

predictable quadratic variation is defined for any locally square-integrable local martingale M with initial value zero, is denoted $\langle M \rangle$, and is the unique predictable, increasing and locally integrable process with initial value zero such that $[M] - \langle M \rangle$ is a local martingale, see p. 124 of [11]. For a continuous local martingale M with initial value zero, we have that M always is locally square integrable and $\langle M \rangle = [M]$.

Using the predictable quadratic variation, the following result is demonstrated in Theorem 9 of [10]. Let M be a locally square integrable local martingale with initial value zero and $\Delta M \geq -1$. It then holds that if $\exp(\frac{1}{2}\langle M^c \rangle_\infty + \langle M^d \rangle_\infty)$ is integrable, then $\mathcal{E}(M)$ is a uniformly integrable martingale. This is an extension of the classical Novikov criterion of [8] to the case with jumps. [10] also argue in Example 10 that the constants in front of $\langle M^c \rangle$ and $\langle M^d \rangle$ are optimal, although their argument contains a flaw, namely that the formula (28) in that paper does not hold.

In this paper, we specialize our efforts to the case where M has jumps larger than or equal to a for some $a \geq -1$ and prove results of the same type, requiring either that M is locally square integrable and that $\exp(\frac{1}{2}\langle M^c \rangle + \alpha(a)\langle M^d \rangle)$ is integrable for some $\alpha(a)$, or that $\exp(\frac{1}{2}[M] + \beta(a)[M^d])$ is integrable for some $\beta(a)$. For all $a \geq -1$, we identify the optimal value of $\alpha(a)$ and $\beta(a)$, in particular giving an argument circumventing the problems of Example 11 in [10]. Our results are stated as Theorem 2.4 and Theorem 2.5. In particular, we obtain that for local martingales M with initial value zero and $\Delta M \geq 0$, $\mathcal{E}(M)$ is a uniformly integrable martingale if $\exp(\frac{1}{2}[M]_\infty)$ is integrable or if M is locally square integrable and $\exp(\frac{1}{2}\langle M \rangle_\infty)$ is integrable, and we obtain that both the constants in the exponents and the requirement on the jumps of M are optimal. This result is stated as Corollary 2.6 and yields a verbatim extension of the Novikov criterion to local martingales M with initial value zero and $\Delta M \geq 0$.

2. Main results and proofs

For $a > -1$ with $a \neq 0$, we define

$$(2.1) \quad \alpha(a) = \frac{(1+a)\log(1+a) - a}{a^2} \quad \text{and}$$

$$(2.2) \quad \beta(a) = \frac{(1+a)\log(1+a) - a}{a^2(1+a)},$$

and put $\alpha(0) = \beta(0) = \frac{1}{2}$ and $\alpha(-1) = 1$. The functions α and β will yield the optimal constants in the criteria we will be demonstrating. Before proving our main results, Theorem 2.4 and Theorem 2.5, we state three lemmas.

Lemma 2.1. *The functions α and β are continuous, positive and strictly decreasing. Furthermore, $\beta(a)$ tends to infinity as a tends to minus one.*

Proof. We first prove the result on α . Define $h(a) = (1+a)\log(1+a) - a$ for $a > -1$ and $h(-1) = 1$. Note that h is differentiable with $h'(a) = \log(1+a)$. By the l'Hôpital rule, we have

$$(2.3) \quad \lim_{a \rightarrow -1} h(a) = 1 + \lim_{a \rightarrow -1} \frac{\log(1+a)}{(1+a)^{-1}} = 1 - \lim_{a \rightarrow -1} \frac{(1+a)^{-1}}{(1+a)^{-2}} = 1,$$

which yields that h and α are continuous at -1 . Similarly,

$$(2.4) \quad \lim_{a \rightarrow 0} \alpha(a) = \lim_{a \rightarrow 0} \frac{\log(1+a)}{2a} = \lim_{a \rightarrow 0} \frac{1}{2(1+a)} = \frac{1}{2},$$

so α is continuous at 0. As h is zero at zero, $h(a)$ is positive for $a \neq 0$, from which it follows that α is positive. It remains to show that α is strictly decreasing. For $a \geq -1$ with $a \notin \{-1, 0\}$, we have that α is differentiable with

$$(2.5) \quad \begin{aligned} \alpha'(a) &= \frac{a^2 \log(1+a) - 2((1+a)\log(1+a) - a)a}{a^4} \\ &= \frac{2a^2 - a(2+a)\log(1+a)}{a^4}. \end{aligned}$$

By the l'Hôpital rule, we obtain

$$(2.6) \quad \begin{aligned} \lim_{a \rightarrow 0} \alpha'(a) &= \lim_{a \rightarrow 0} \frac{4a - 2(1+a)\log(1+a) - a(2+a)(1+a)^{-1}}{4a^3} \\ &= \lim_{a \rightarrow 0} \frac{a(2+a)(1+a)^{-2} - 2\log(1+a)}{12a^2} \\ &= -\lim_{a \rightarrow 0} \frac{2a(2+a)(1+a)^{-3}}{24a} = -\frac{1}{12} \lim_{a \rightarrow 0} \frac{2+a}{(1+a)^3} = -\frac{1}{6}, \end{aligned}$$

so defining $\alpha'(0) = -\frac{1}{6}$, we obtain that α' is a continuous mapping on $(-1, \infty)$, and as α' is the derivative of α for $a \geq -1$ with $a \notin \{-1, 0\}$, α' is also the derivative of α for $(1, \infty)$. In order to show that α is strictly decreasing, it then suffices to show that $2a^2 - a(2+a)\log(1+a)$ is negative for $a > -1$ with $a \neq 0$. Now, for $a \neq 0$, note that

$$(2.7) \quad \frac{d}{da}(2a - (2+a)\log(1+a)) = 2 - \log(1+a) - \frac{2+a}{1+a} \quad \text{and}$$

$$(2.8) \quad \frac{d^2}{da^2}(2a - (2+a)\log(1+a)) = \frac{1}{(1+a)^2} - \frac{1}{1+a} = -\frac{a}{(1+a)^2}.$$

From this, we conclude that $a \mapsto 2a - (2+a)\log(1+a)$ is positive for $-1 < a < 0$ and negative for $a > 0$. Therefore, $a \mapsto 2a^2 - a(2+a)\log(1+a)$ is negative for $a > -1$ with $a \neq 0$. As a consequence, α is strictly decreasing. As $\beta(a) = \alpha(a)/(1+a)$, the results on β follow from those on α . \square

Lemma 2.2. *Let N be a standard Poisson process, let b and λ be in \mathbb{R} , and define $f_b(\lambda) = \exp(-\lambda) + \lambda(1+b) - 1$. With $L_t^b = \exp(-\lambda(N_t - (1+b)t) - tf_b(\lambda))$, L^b is a nonnegative martingale with respect to the filtration induced by N .*

Proof. Let $\mathcal{G}_t = \sigma(N_s)_{s \leq t}$. Fix $0 \leq s \leq t$. As $N_t - N_s$ is independent of \mathcal{G}_s and follows a Poisson distribution with parameter $t - s$, we obtain

$$(2.9) \quad E(\exp(-\lambda(N_t - N_s)) | \mathcal{G}_s) = \exp((t - s)(\exp(-\lambda) - 1)),$$

which implies

$$\begin{aligned} E(L_t^b | \mathcal{G}_s) &= E(\exp(-\lambda(N_t - N_s)) | \mathcal{G}_s) \exp(-\lambda N_s) \exp(\lambda(1 + b)t - t f_b(\lambda)) \\ &= \exp((t - s)(\exp(-\lambda) - 1)) \exp(-\lambda N_s) \exp(\lambda(1 + b)t - t f_b(\lambda)) \\ (2.10) \quad &= \exp(-\lambda(N_s - (1 + b)s) - s f_b(\lambda)) = L_s^b, \end{aligned}$$

proving the lemma. \square

Lemma 2.3. *Let M be a local martingale with initial value zero and $\Delta M \geq -1$. Then $E\mathcal{E}(M)_\infty \leq 1$, and $\mathcal{E}(M)$ is a uniformly integrable martingale if and only if $E\mathcal{E}(M)_\infty = 1$.*

Proof. This follows from the optional sampling theorem for nonnegative supermartingales. \square

In the proof of Theorem 2.4, note that for a standard Poisson process N , it holds that with $M_t = N_t - t$, $\langle M \rangle_t = t$, since $[M]_t = N_t$ by Definition VI.37.6 of [13] and since $\langle M \rangle$ is the unique predictable and locally integrable increasing process making $[M] - \langle M \rangle$ a local martingale.

Theorem 2.4. *Fix $a \geq -1$. Let M be a locally square integrable local martingale with $\Delta M 1_{(\Delta M \neq 0)} \geq a$. If $\exp(\frac{1}{2}\langle M^c \rangle_\infty + \alpha(a)\langle M^d \rangle_\infty)$ is integrable, then $\mathcal{E}(M)$ is a uniformly integrable martingale. Furthermore, for all $a \geq -1$, the coefficients $\frac{1}{2}$ and $\alpha(a)$ in front of $\langle M^c \rangle$ and $\langle M^d \rangle$ are optimal in the sense that the criterion is false if any of the coefficients are reduced.*

Proof. Sufficiency. With $h(x) = (1 + x)\log(1 + x) - x$, we find by Lemma 2.1 that for $-1 \leq a \leq x$, $\alpha(a) \geq \alpha(x)$, which implies $h(x) \leq \alpha(a)x^2$. Letting $a \geq -1$ and letting M be a locally square integrable local martingale with initial value zero, $\Delta M 1_{(\Delta M \neq 0)} \geq a$ and $\exp(\frac{1}{2}\langle M^c \rangle_\infty + \alpha(a)\langle M^d \rangle_\infty)$ integrable, we obtain for all $t \geq 0$ the inequality $(1 + \Delta M_t)\log(1 + \Delta M_t) + \Delta M_t \leq \alpha(a)(\Delta M_t)^2$, and so Theorem III.1 of [7] shows that $\mathcal{E}(M)$ is a uniformly integrable martingale. Thus, the condition is sufficient.

As regards optimality of the coefficients, optimality of the coefficient $\frac{1}{2}$ in front of $\langle M^c \rangle$ is well-known, see [8]. It therefore suffices to prove optimality of the coefficient $\alpha(a)$ in front of $\langle M^d \rangle$. To do so, we need to show the following: That for each $\varepsilon > 0$, there exists a locally square integrable local martingale with initial value zero and $\Delta M \geq a$ such that $\exp(\frac{1}{2}\langle M^c \rangle_\infty + (1 - \varepsilon)\alpha(a)\langle M^d \rangle_\infty)$ is integrable, while $\mathcal{E}(M)$ is not a uniformly integrable martingale.

The case $a > 0$. Let $\varepsilon, b > 0$, put $T_b = \inf\{t \geq 0 \mid N_t - (1+b)t = -1\}$ and define $M_t = a(N_t^{T_b} - t \wedge T_b)$. We claim that we may choose $b > 0$ such that M satisfies the requirements stated above. It holds that M is a locally square integrable local martingale with initial value zero and $\Delta M 1_{(\Delta M \neq 0)} \geq a$, and M is purely discontinuous by Definition 7.21 of [2] since it is of locally integrable variation. In particular, $M^c = 0$, so it suffices to show that $\exp((1-\varepsilon)\alpha(a)\langle M \rangle_\infty)$ is integrable while $\mathcal{E}(M)$ is not a uniformly integrable martingale. To show this, we first argue that T_b is almost surely finite. To this end, note that since $t \mapsto N_t - (1+b)t$ only has nonnegative jumps, has initial value zero and decreases between jumps, the process hits -1 if and only if it is less than or equal to -1 immediately before one of its jumps. Therefore, with U_n denoting the n 'th jump time of N , we have

$$\begin{aligned} P(T_b = \infty) &= P(\cap_{n=1}^\infty (N_{U_n-} - (1+b)U_n > -1)) \\ &= P(\cap_{n=1}^\infty (n > U_n(1+b))) \\ (2.11) \quad &\leq P(\limsup_{n \rightarrow \infty} U_n/n \leq (1+b)^{-1}), \end{aligned}$$

which is zero, as $\lim_{n \rightarrow \infty} U_n/n = 1$ almost surely by the law of large numbers, and $(1+b)^{-1} < 1$ as $b > 0$. Thus, T_b is almost surely finite, and by the path properties of N , $N_{T_b} = (1+b)T_b - 1$ almost surely. We then obtain

$$\begin{aligned} \mathcal{E}(M)_\infty &= \exp(a(N_{T_b} - T_b) + N_{T_b}(\log(1+a) - a)) \\ &= \exp(N_{T_b} \log(1+a) - aT_b) \\ &= \exp(((1+b)T_b - 1) \log(1+a) - aT_b) \\ (2.12) \quad &= (1+a)^{-1} \exp(T_b((1+b) \log(1+a) - a)). \end{aligned}$$

Recalling Lemma 2.3, we wish to choose $b > 0$ such that $E\mathcal{E}(M)_\infty < 1$ and $E \exp((1-\varepsilon)\alpha\langle M \rangle_\infty) < \infty$ holds simultaneously. Note that $\langle M \rangle_\infty = a^2 T_b$. Therefore, we need to select a positive b with the properties that

$$(2.13) \quad E \exp(T_b((1+b) \log(1+a) - a)) < 1+a \text{ and}$$

$$(2.14) \quad E \exp(T_b a^2(1-\varepsilon)\alpha(a)) < \infty.$$

Consider some $b > 0$ and let f_b be as in Lemma 2.2. By that same lemma, the process L^b defined by putting $L_t^b = \exp(-\lambda(N_t - (1+b)t) - t f_b(\lambda))$ is a martingale. In particular, it is a nonnegative supermartingale with initial value one, so Theorem II.77.5 of [12] yields $1 \geq EL_{T_b}^b = E \exp(\lambda - T_b f_b(\lambda))$, and so $E \exp(-T_b f_b(\lambda)) \leq \exp(-\lambda)$. Note that $f'_b(\lambda) = -\exp(-\lambda) + 1+b$, such that f_b takes its minimum at $-\log(1+b)$. Therefore, $-f_b$ takes its maximum at $-\log(1+b)$, and we find that the maximum is $h(b)$. In particular, $E \exp(T_b h(b))$ is finite. Next, define a function λ by putting $\lambda(b) = -\log((1+a)\frac{b}{a})$, we then have $E \exp(-T_b f_b(\lambda(b))) \leq (1+a)\frac{b}{a}$, which is strictly less than $1+a$ whenever $b < a$. Thus, if we can choose $b \in (0, a)$ such that

$$(2.15) \quad (1+b) \log(1+a) - a \leq -f_b(\lambda(b)) \text{ and}$$

$$(2.16) \quad a^2(1-\varepsilon)\alpha(a) \leq h(b),$$

we will have achieved our end, since (2.15) implies (2.13) and (2.16) implies (2.14). To this end, note that

$$\begin{aligned}
 -f_b(\lambda(b)) &= -\exp(\log((1+a)\frac{b}{a})) + \log((1+a)\frac{b}{a})(1+b) + 1 \\
 &= 1 - (1+a)\frac{b}{a} + (1+b)\log((1+a)\frac{b}{a}) \\
 (2.17) \quad &= 1 - (1+a)\frac{b}{a} + (1+b)\log(1+a) + (1+b)\log\frac{b}{a},
 \end{aligned}$$

such that, by rearrangement, (2.15) is equivalent to

$$(2.18) \quad 0 \leq 1 + a - (1+a)\frac{b}{a} + (1+b)\log\frac{b}{a},$$

and therefore, as $1 - \frac{b}{a} > 0$ for $0 < b < a$, equivalent to

$$(2.19) \quad (1+b)\frac{\log\frac{b}{a}}{\frac{b}{a}-1} \leq 1+a,$$

which, as $\log x \leq x - 1$ for $x > 0$, is satisfied for all $0 < b < a$. Thus, it suffices to choose $b \in (0, a)$ such that (2.16) is satisfied, corresponding to choosing $b \in (0, a)$ such that $(1 - \varepsilon)h(a) \leq h(b)$. As h is positive and continuous on $(0, \infty)$, this is possible by choosing b close enough to a . With this choice of b , we now obtain M yielding an example proving that the coefficient $\alpha(a)$ is optimal. This concludes the proof of optimality in the case $a > 0$.

The case $a = 0$. Let $\varepsilon > 0$. To prove optimality, we wish to identify a locally square integrable local martingale M with initial value zero and $\Delta M 1_{(\Delta M \neq 0)} \geq 0$ such that $\exp((1 - \varepsilon)\alpha(0)\langle M \rangle_\infty)$ is integrable while $\mathcal{E}(M)$ is not a uniformly integrable martingale. Recalling that α is positive and continuous, pick $a > 0$ so close to zero that $(1 - \varepsilon)\alpha(0) \leq (1 - \frac{1}{2}\varepsilon)\alpha(a)$. By what was already shown, there exists a locally square integrable local martingale M with initial value zero and $\Delta M 1_{(\Delta M \neq 0)} \geq a$ such that $\exp((1 - \frac{1}{2}\varepsilon)\alpha(a)\langle M \rangle_\infty)$ is integrable while $\mathcal{E}(M)$ is not a uniformly integrable martingale. As $\exp((1 - \varepsilon)\alpha(0)\langle M \rangle_\infty)$ is integrable in this case, this shows that $\alpha(0)$ is optimal.

The case $-1 < a < 0$. Let $\varepsilon > 0$, let $-1 < b < 0$, let $c > 0$ and define a stopping time T_{bc} by putting $T_{bc} = \inf\{t \geq 0 \mid N_t - (1+b)t \geq c\}$. Also define M by $M_t = a(N_t^{T_{bc}} - t \wedge T_{bc})$. We claim that we can choose $b \in (-1, 0)$ and $c > 0$ such that M satisfies the requirements to show optimality. Similarly to the case $a > 0$, M is a purely discontinuous locally square integrable local martingale with initial value zero and $\Delta M 1_{(\Delta M \neq 0)} \geq a$, so it suffices to show that $\exp((1 - \varepsilon)\alpha(a)\langle M \rangle_\infty)$ is integrable while $\mathcal{E}(M)$ is not a uniformly integrable martingale. We first investigate some properties of T_{bc} . As $t \mapsto N_t - (1+b)t$ only has nonnegative jumps, has initial value zero and decreases between jumps, the process advances beyond c at some point if and only if it advances beyond c at one of its jump times. Therefore,

with U_n denoting the n 'th jump time of N ,

$$\begin{aligned}
 P(T_{bc} = \infty) &= P(\cap_{n=1}^{\infty} (N_{U_n} - (1+b)U_n < c)) \\
 &= P(\cap_{n=1}^{\infty} (n - c < U_n(1+b))) \\
 (2.20) \quad &\leq P(\liminf_{n \rightarrow \infty} U_n/n \geq (1+b)^{-1}),
 \end{aligned}$$

which is zero, as U_n/n tends to one almost surely and $(1+b)^{-1} > 1$. Thus, T_{bc} is almost surely finite. Furthermore, by the path properties of N , $N_{T_{bc}} \geq (1+b)T_{bc} + c$ and $N_{T_{bc}} \leq (1+b)T_{bc} + c + 1$ almost surely. Since $\log(1+a) \leq 0$, we in particular obtain $N_{T_{bc}} \log(1+a) \leq ((1+b)T_{bc} + c) \log(1+a)$ almost surely. From this, we conclude that

$$\begin{aligned}
 \mathcal{E}(M)_{\infty} &= \exp(a(N_{T_{bc}} - T_{bc}) + N_{T_{bc}}(\log(1+a) - a)) \\
 &= \exp(N_{T_{bc}} \log(1+a) - aT_{bc}) \\
 &\leq \exp(((1+b)T_{bc} + c) \log(1+a) - aT_{bc}) \\
 (2.21) \quad &= (1+a)^c \exp(T_{bc}((1+b) \log(1+a) - a)).
 \end{aligned}$$

We wish to choose $-1 < b < 0$ and $c > 0$ such that $E \exp((1-\varepsilon)\alpha(a)\langle M \rangle_{\infty}) < \infty$ and $E \mathcal{E}(M)_{\infty} < 1$ holds simultaneously. As $\langle M \rangle_{\infty} = a^2 T_{bc}$, this is equivalent to choosing $-1 < b < 0$ and $c > 0$ such that

$$(2.22) \quad E \exp(T_{bc}((1+b) \log(1+a) - a)) < (1+a)^{-c} \text{ and}$$

$$(2.23) \quad E \exp(T_{bc} a^2 (1-\varepsilon)\alpha(a)) < \infty.$$

Let f_b and L^b be as in Lemma 2.2. The process L^b is then a nonnegative supermartingale. As $N_{T_{bc}} \leq (1+b)T_{bc} + c + 1$, the optional stopping theorem allows us to conclude that for $\lambda \geq 0$,

$$\begin{aligned}
 1 &\geq E L_{T_{bc}}^b = E \exp(-\lambda(N_{T_{bc}} - (1+b)T_{bc}) - T_{bc} f_b(\lambda)) \\
 (2.24) \quad &\geq E \exp(-(c+1)\lambda - T_{bc} f_b(\lambda)),
 \end{aligned}$$

so that $E \exp(-T_{bc} f_b(\lambda)) \leq \exp((c+1)\lambda)$. As in the case $a > 0$, $-f_b$ takes its maximum at $-\log(1+b)$, and the maximum is $h(b)$, leading us to conclude that $E \exp(T_{bc} h(b))$ is finite. Put $\lambda(b, c) = (c+1)^{-1} \log((1+a)^{-c} \frac{b}{a})$. For all $b \in (a, 0)$, $\frac{b}{a} < 1$, leading to $E \exp(-T_{bc} f_b(\lambda(b, c))) \leq (1+a)^{-c} \frac{b}{a} < (1+a)^{-c}$. Therefore, if we can choose $b \in (a, 0)$ and $c > 0$ such that

$$(2.25) \quad (1+b) \log(1+a) - a \leq -f_b(\lambda(b, c)) \text{ and}$$

$$(2.26) \quad a^2(1-\varepsilon)\alpha(a) \leq h(b),$$

we will have obtained existence of a local martingale yielding the desired optimality of $\alpha(a)$. We first note that $a^2(1-\varepsilon)\alpha(a) \leq h(b)$ is equivalent to $(1-\varepsilon)h(a) \leq h(b)$. As h is continuous and positive on $(-1, 0)$, we find that (2.26) is satisfied for $a < b < 0$ with b close enough to a . Next, we turn our attention to (2.25). We

have

$$\begin{aligned}
-f_b(\lambda(b, c)) &= -\exp\left(-\frac{1}{c+1}\log\left((1+a)^{-c}\frac{b}{a}\right)\right) - \frac{1+b}{c+1}\log\left((1+a)^{-c}\frac{b}{a}\right) + 1 \\
&= 1 - (1+a)^{\frac{c}{c+1}}\left(\frac{b}{a}\right)^{-\frac{1}{c+1}} - \frac{1+b}{c+1}\log\left((1+a)^{-c}\frac{b}{a}\right) \\
(2.27) \quad &= 1 - (1+a)^{\frac{c}{c+1}}\left(\frac{a}{b}\right)^{\frac{1}{c+1}} + \frac{c(1+b)}{c+1}\log(1+a) + \frac{1+b}{c+1}\log\frac{a}{b},
\end{aligned}$$

such that (2.25) is equivalent to

$$(2.28) \quad 0 \leq 1 + a - (1+a)^{\frac{c}{c+1}}\left(\frac{a}{b}\right)^{\frac{1}{c+1}} + \frac{1+b}{c+1}\left(\log\frac{a}{b} - \log(1+a)\right).$$

Fixing $a < b < 0$, we wish to argue that for b close enough to a , (2.28) holds for c large enough. To this end, let $\rho_b(c)$ denote the right-hand side of (2.28). Then $\lim_{c \rightarrow \infty} \rho_b(c) = 0$. We also note that $\frac{d}{dc} \frac{1}{c+1} = -\frac{1}{(c+1)^2}$ and $\frac{d}{dc} \frac{c}{c+1} = \frac{1}{(c+1)^2}$, yielding

$$\begin{aligned}
&\frac{d}{dc}(1+a)^{\frac{c}{c+1}}\left(\frac{a}{b}\right)^{\frac{1}{c+1}} \\
&= \frac{d}{dc} \exp\left(\frac{c}{c+1}\log(1+a) + \frac{1}{c+1}\log\frac{a}{b}\right) \\
&= \left(\frac{\log(1+a)}{(c+1)^2} - \frac{\log\frac{a}{b}}{(c+1)^2}\right) \exp\left(\frac{c}{c+1}\log(1+a) + \frac{1}{c+1}\log\frac{a}{b}\right) \\
(2.29) \quad &= \frac{\log(1+a) - \log\frac{a}{b}}{(c+1)^2} \exp\left(\frac{c}{c+1}\log(1+a) + \frac{1}{c+1}\log\frac{a}{b}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{d}{dc} \frac{1+b}{c+1} \left(\log\frac{a}{b} - \log(1+a)\right) = -\frac{1+b}{(c+1)^2} \left(\log\frac{a}{b} - \log(1+a)\right) \\
(2.30) \quad &= (1+b) \frac{\log(1+a) - \log\frac{a}{b}}{(c+1)^2},
\end{aligned}$$

which leads to

$$(2.31) \quad \rho'_b(c) = \frac{\log(1+a) - \log\frac{a}{b}}{(c+1)^2} \left(1+b - \exp\left(\frac{c}{c+1}\log(1+a) + \frac{1}{c+1}\log\frac{a}{b}\right)\right).$$

Now note that for $a < b$, we obtain

$$(2.32) \quad \lim_{c \rightarrow \infty} 1+b - \exp\left(\frac{c}{c+1}\log(1+a) + \frac{1}{c+1}\log\frac{a}{b}\right) = 1+b - (1+a) > 0,$$

and for b close enough to a , $\log(1+a) - \log\frac{a}{b} < 0$, since $a < 0$. Therefore, for all c large enough, $\rho'_b(c) < 0$. Consider such a c , we then obtain

$$(2.33) \quad \rho_b(c) = \lim_{y \rightarrow \infty} \rho_b(c) - \rho_b(y) = -\lim_{y \rightarrow \infty} \int_c^y \rho'_b(z) dz > 0.$$

Thus, we conclude that for b close enough to a , it holds that $\rho_b(c) > 0$ for c large enough.

We now collect our conclusions in order to obtain $b \in (a, 0)$ and $c > 0$ satisfying (2.25) and (2.26). First choose $b \in (a, 0)$ so close to a that $(1 - \varepsilon)h(a) \leq h(b)$ and $\log(1 + a) - \log \frac{a}{b} < 0$. Pick c so large that $\rho_b(c) > 0$. By our deliberations, (2.25) and (2.26) then both hold, demonstrating the existence of a locally square integrable local martingale M with initial value zero and $\Delta M 1_{(\Delta M \neq 0)} \geq a$ such that $\exp((1 - \varepsilon)\alpha(a)\langle M \rangle_\infty)$ is integrable while $\mathcal{E}(M)$ is not a uniformly integrable martingale.

The case $a = -1$. Let $\varepsilon > 0$. We wish to identify a purely discontinuous locally square integrable local martingale M with $\Delta M 1_{(\Delta M \neq 0)} \geq -1$ such that integrability of $\exp((1 - \varepsilon)\alpha(-1)\langle M \rangle_\infty)$ holds while $\mathcal{E}(M)$ is not a uniformly integrable martingale. We proceed as in the case $a = 0$. By positivity and continuity of α , take $a > 0$ so close to -1 that $(1 - \varepsilon)\alpha(-1) \leq (1 - \frac{1}{2}\varepsilon)\alpha(a)$. By what was shown in the previous case, there exists a purely discontinuous locally square integrable local martingale M with initial value zero and $\Delta M 1_{(\Delta M \neq 0)} \geq a$ such that $\exp((1 - \frac{1}{2}\varepsilon)\alpha(a)\langle M \rangle_\infty)$ is integrable while $\mathcal{E}(M)$ is not a uniformly integrable martingale. As $\exp((1 - \varepsilon)\alpha(-1)\langle M \rangle_\infty)$ then also is integrable, this shows that $\alpha(-1)$ is optimal. \square

Theorem 2.5. *Fix $a > -1$. Let M be a local martingale with $\Delta M 1_{(\Delta M \neq 0)} \geq a$. If $\exp(\frac{1}{2}[M^c]_\infty + \beta(a)[M^d]_\infty)$ is integrable, then $\mathcal{E}(M)$ is a uniformly integrable martingale. Furthermore, for all $a > -1$, the coefficients $\frac{1}{2}$ and $\beta(a)$ in front of $[M^c]$ and $[M^d]$ are optimal in the sense that the criterion is false if any of the coefficients are reduced.*

Furthermore, there exists no $\beta(-1)$ such that for M with $\Delta M 1_{(\Delta M \neq 0)} \geq -1$, integrability of $\exp(\frac{1}{2}[M^c]_\infty + \beta(-1)[M^d]_\infty)$ suffices to ensure that $\mathcal{E}(M)$ is a uniformly integrable martingale.

Proof. Sufficiency. We proceed in a manner closely related to the proof of Theorem 2.4. Defining g by putting $g(x) = \log(1 + x) - x/(1 + x)$, we find by Lemma 2.1 that for $-1 < a \leq x$, $\beta(a) \geq \beta(x)$, yielding $g(x) \leq \beta(a)x^2$. Letting $a > -1$ and letting M be a locally square integrable local martingale with initial value zero, $\Delta M 1_{(\Delta M \neq 0)} \geq a$ and $\exp(\frac{1}{2}\langle M^c \rangle_\infty + \beta(a)\langle M^d \rangle_\infty)$ integrable, we obtain for all $t \geq 0$ that $\log(1 + \Delta M_t) + \Delta M_t/(1 + \Delta M_t) \leq \beta(a)(\Delta M_t)^2$, and so Theorem III.7 of [7] shows that $\mathcal{E}(M)$ is a uniformly integrable martingale. Thus, the condition is sufficient.

As in Theorem 2.4, optimality of the $\frac{1}{2}$ in front of $[M^c]$ follows from [8], so it suffices to consider the coefficient $\beta(a)$ in front of $[M^d]$. Thus, for $a > -1$, we need to prove that for each $\varepsilon > 0$, there exists a locally square integrable local martingale with initial value zero and $\Delta M 1_{(\Delta M \neq 0)} \geq a$ such that $\exp(\frac{1}{2}[M^c]_\infty + (1 - \varepsilon)\beta(a)[M^d]_\infty)$ is integrable, while $\mathcal{E}(M)$ is not a uniformly integrable martingale.

The case $a > 0$. Let $\varepsilon, b > 0$, put $T_b = \inf\{t \geq 0 \mid N_t - (1+b)t = -1\}$ and define $M_t = a(N_t^{T_b} - t \wedge T_b)$. Noting that $[M]_\infty = a^2 N_{T_b}$, we may argue as in the proof of Theorem 2.4 and obtain that it suffices to identify $b > 0$ such that

$$(2.34) \quad E \exp(T_b((1+b) \log(1+a) - a)) < 1+a \text{ and}$$

$$(2.35) \quad E \exp(N_{T_b} a^2(1-\varepsilon)\beta(a)) < \infty.$$

Let f_b be as in Lemma 2.2. As in the proof of Theorem 2.4, we obtain that $E \exp(T_b h(b))$ is finite, where $h(x) = (1+x) \log(1+x) - x$, and furthermore obtain that with $\lambda(b) = -\log((1+a)\frac{b}{a})$, $E \exp(-T_b f_b(\lambda(b))) < 1+a$ for $b < a$. As $N_{T_b} = (1+b)T_b - 1$ almost surely and $g(b) = h(b)/(1+b)$, we then also obtain that $E \exp(N_{T_b} g(b))$ is finite. Thus, if we can choose $b \in (0, a)$ such that

$$(2.36) \quad (1+b) \log(1+a) - a \leq -f_b(\lambda(b)) \text{ and}$$

$$(2.37) \quad a^2(1-\varepsilon)\beta(a) \leq g(b),$$

we will obtain the desired result, as (2.36) implies (2.34) and (2.37) implies (2.35). As earlier noted, (2.36) always holds for $0 < b < a$. As for (2.37), this requirement is equivalent to having that $(1-\varepsilon)g(a) \leq g(b)$ for some $b \in (0, a)$, which by continuity of g can be obtained by choosing b close enough to a . Choosing b in this manner, we obtain M yielding an example proving that the coefficient $\beta(a)$ is optimal. This concludes the proof of optimality in the case $a > 0$.

The case $a = 0$. This follows similarly to the corresponding case in the proof of Theorem 2.4.

The case $-1 < a < 0$. Let $\varepsilon > 0$, let $-1 < b < 0$, let $c > 0$ and define a stopping time T_{bc} by putting $T_{bc} = \inf\{t \geq 0 \mid N_t - (1+b)t \geq c\}$. Also define M by $M_t = a(N_t^{T_{bc}} - t \wedge T_{bc})$. As in the proof of Theorem 2.4, in order to obtain the desired counterexample, it suffices to choose $-1 < b < 0$ and $c > 0$ such that

$$(2.38) \quad E \exp(T_{bc}((1+b) \log(1+a) - a)) < (1+a)^{-c} \text{ and}$$

$$(2.39) \quad E \exp(T_{bc} a^2(1-\varepsilon)\beta(a)) < \infty.$$

With f_b as in Lemma 2.2, we find as in the proof of Theorem 2.4 that $E \exp(T_{bc} h(b))$ is finite. Furthermore, defining $\lambda(b, c) = (c+1)^{-1} \log((1+a)^{-c} \frac{b}{a})$, it holds for b with $a < b \leq (1+a)^c a$ that $\lambda(b, c) \geq 0$ and $E \exp(-T_{bc} f_b(\lambda(b, c))) < (1+a)^{-c}$. Also, as $N_{T_{bc}} \leq (1+b)T_{bc} + c + 1$, $E \exp(N_{T_{bc}}(1+b)^{-1}h(b))$ and thus $E \exp(N_{T_{bc}} g(b))$ is finite. Therefore, if we can choose $b \in (a, 0)$ and $c > 0$ such that

$$(2.40) \quad (1+b) \log(1+a) - a \leq -f_b(\lambda(b, c)) \text{ and}$$

$$(2.41) \quad a^2(1-\varepsilon)\beta(a) \leq g(b),$$

we obtain the desired result. By arguments as in the proof of the corresponding case of Theorem 2.4, we find that by first picking b close enough to a and then c large enough, we can ensure that both (2.40) and (2.41) hold, yielding optimality for this case.

The case $a = -1$. For this case, we need to show that for any $\gamma \geq 0$, it does not hold that finiteness of $E \exp(\gamma[M^d]_\infty)$ implies that $\mathcal{E}(M)$ is a uniformly integrable martingale. Let $\gamma \geq 0$. By Lemma 2.1, $\beta(a)$ tends to infinity as a tends to -1 . Therefore, we may pick $a > -1$ so small that $\beta(a) \geq \gamma$. By what we already have shown, there exists M with initial value zero and $\Delta M 1_{(\Delta M \neq 0)} \geq -1$ such that $E \exp(\beta(a)[M^d]_\infty)$ and thus $E \exp(\gamma[M^d]_\infty)$ is finite, while $\mathcal{E}(M)$ is not a uniformly integrable martingale. \square

Corollary 2.6. *Let M be a local martingale with initial value zero and $\Delta M \geq 0$. If $\exp(\frac{1}{2}[M]_\infty)$ is integrable or if M is locally square integrable and $\exp(\frac{1}{2}\langle M \rangle_\infty)$ is integrable, then $\mathcal{E}(M)$ is a uniformly integrable martingale. Furthermore, this criterion is optimal in the sense that if either the constant $\frac{1}{2}$ is reduced, or the requirement on the jumps is weakened to $\Delta M \geq -\varepsilon$ for some $\varepsilon > 0$, the criterion ceases to be sufficient.*

Proof. That the constant $\frac{1}{2}$ cannot be reduced follows from Theorem 2.4 and Theorem 2.5. That the requirement on the jumps cannot be reduced follows by combining Theorem 2.4 and Theorem 2.5 with the fact that α and β both are strictly decreasing by Lemma 2.1. \square

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